The Board of

red'shine
PUBLICATION

Is hereby Awarding this Certificate to

Somnath Shivram Sanap

In recognition of the publication of the chapter

" COMPARISON OF DIFFERENTIABILITY IN R AND C "

In the peer-reviewed edited book entitled
"MULTIDISCIPLINARY APPROACH IN RESEARCH, VOLUME-20"
Edited by Dr. Dilipkumar A. Ode, Dr. Lalit Upadhyay, Zaheen Hasan, Rachana Singh, Somnath Shivram Sanap, Akashdeep Barman, Dr. Machhindranath M Dhane, Dr. Nasirahmed M Jangubhai

Published in Category of Edited Book
December- 2022(First Edition)
ISBN: 978-91-988207-0-6 | ISBN-10: 91-988207-0-2 | DIP: 10.25215/9198820702 | DOI: 10.25215/9198820702
www.redshine.co.in

Dilip. A. ode

DR. DILIPKUMAR A. ODE
Assistant Professor, Department of Economics
C & S. H. Desai Arts & L. K. L. Doshi Commerce College, Balasinor (Gujarat), India

ificate Number 81385753328

REDSHINE CERTIFIED

ISBN: 978-91-988207-0-6

P: 18.10.9198820702.003

COMPARISON OF DIFFERENTIABILITY IN R AND C

SOMNATH SHIVRAM SANAP
Assistant Professor
Department of Mathematics
A V College of Arts KM College of Commerce and ESA College of Science
Vasai Road, Palghar (Maharashtra), India

INTRODUCTION:

The concept of limit, continuity and derivative plays an important role in real and complex analysis. In real analysis $\lim_{x \to a} f(x) = L$ means that value of the function f(x) can be made arbitrarily close to real number L if the values of x are chosen sufficiently close to, but not equal to the real number a. The complex limit is analogous to the concept of limit in real analysis in the sense that $\lim_{z \to z_0} f(z) = L$ means that the value of the function $f(z)$ can be made arbitrarily close to the complex number L if values of z are chosen sufficiently close to, but not equal to the complex number z_0. A concept of limit in complex is the same as a real limit except that it is based on notion of "close" in complex plane. As the concept of continuity, derivative was all defined in terms of the concept of a limit, so there is important difference between the concept of derivative of a function in case of calculus of real function $f(x)$ and complex function $f(z)$.In this chapter we will define limit, derivative of a complex function and interpret the difference between the real limit, derivative with the complex limit and derivative with the help of examples.

DEFINITION 1.1 LIMIT OF A COMPLEX FUNCTION:

Let f be a complex valued function defined in a deleted neighborhood of z_0 and suppose that L is a complex number. We say that the "limit of f as z tends to z_0 exist and equal to L" if for every $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and written as
$\lim_{z \to z_0} f(z) = L$ if $0 < |z - z_0| < \delta$ then $|f(z) - L| < \epsilon$

Complex and real limit have many common properties but there is one very important difference.
For real function $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

That is, there are two directions from which x can approach a on the real line (From the right or from the left). The real limit exists if and only if both the limit has equal value. But in case of complex function z is allowed to approach z_0 along infinite number of paths (From any direction in the complex plane). Therefore, in order that $\lim_{z \to z_0} f(z) = L$ exists and equal to L, $f(z)$ must approach the same complex number L along every possible path through z_0.

ISBN: 978-91-988207-0-6

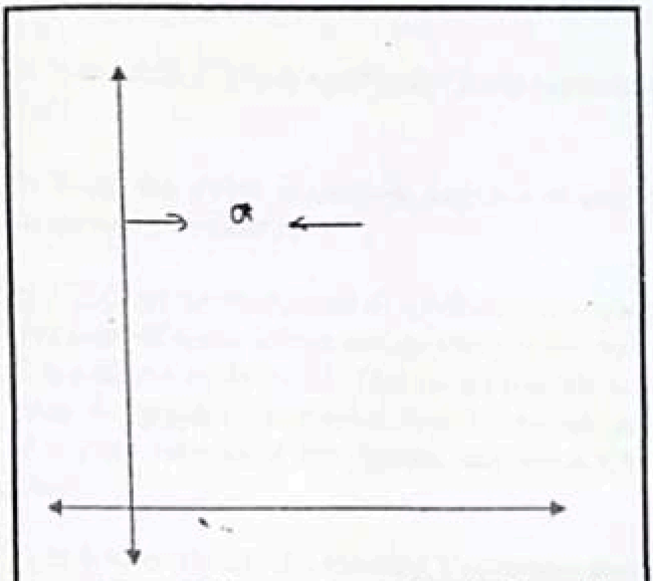

Figure 1.2.1 Two ways to approach a point along real line

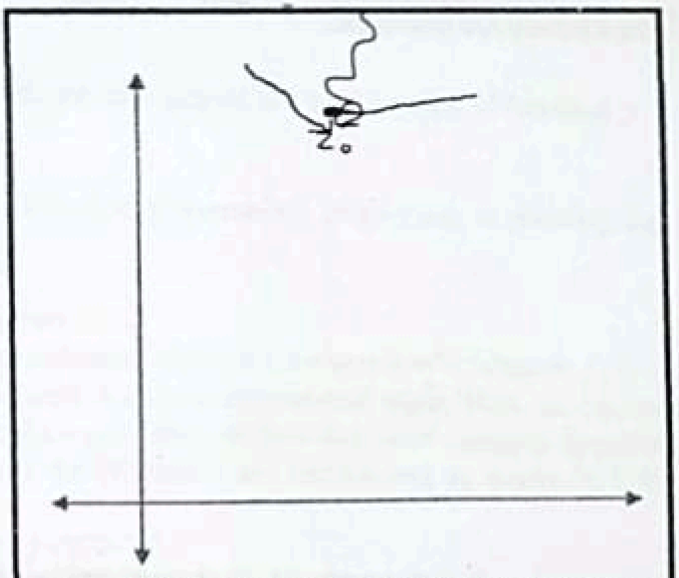

Figure 1.2.2 Different ways to approach point z in complex plane

Definition 1.3 Derivative of a Complex Function

Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0, denoted by $f'(z_0)$ is

$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ provided limit exists ....(i)

Let $\Delta z = z - z_0$

Therefore $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ ....(ii)

For a complex function f to be differentiable at a point z_0, the limit in (ii) must exist and equal to the same complex number from any direction. That is the limit must exist regardless of how Δz approaches 0. This means that, in complex analysis, the requirement of differentiability of a function f(z) at z_0 is a far greater demand than in real calculus of function f(x) where we can approach a real number x_0 along only two directions.

❖ RULES OF DIFFERENTIATION:

All the familiar rules of differentiation i.e., constant rule, sum rule, product rule, quotient rule, chain rule in the calculus of real functions carry over the calculus of complex functions.

Theorem-1.4 If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

Theorem-1.5 Cauchy-Riemann Equations

Suppose $f(z) = u(x,y) + i\, v(x,y)$ is differentiable at a point z_0, then at point z_0 the first order partial derivatives of u and v exist and satisfy the C-R equations

$$u_x = v_y \text{ and } u_y = -v_x$$

❖ DEFINITION 1.6 ANALYTIC FUNCTION:

A complex function f(z) is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

❖ COMPARISON WITH REAL ANALYSIS:

1. In real calculus the derivative of functions y=f(x) at a point has many graphical and physical interpretations.

ISBN: 978-91-988207-0-6

e.g.,

a) Derivative $f'(x)$ at a particular point represent the slope of a tangent line to the graph of function $y = f(x)$

b) When the slope is positive, negative or zero, the function is increasing, decreasing or possibly has maximum or minimum.

c) $f'(x)$ can be interpreted as a velocity of moving object.
But none of these interpretations carry to the complex calculus. However, the graph of a function $f: \mathbb{C} \to \mathbb{C}$ is a subset of $\mathbb{R}^2 \times \mathbb{R}^2$. That is not possible to represent it on two-dimensional page. Thus, we cannot draw the graph of a complex function on one graph. The geometric representation of complex function $w = f(z)$ consist of two figures, the first a subset in the XY-plane and the second its image in UV-plane.

2. In real analysis, if a function f possesses first derivative, there is no guarantee that f possesses any other higher derivatives.
e.g., On interval (-2,2), $f(x) = x^{3/2}$ is differentiable at x=0 but $f'(x) = \frac{3}{2}x^{1/2}$ is not differentiable at x=0.
But in complex analysis, if a function f is analytic in a domain D, this will guarantee that f possesses higher order derivatives at all points in D.

Examples 1.7

1. The real function $f(x) = x$ is differentiable everywhere, but the complex function $f(z) = Re(z)$ Or $f(z) = x$ is nowhere differentiable in the whole complex plane.
We have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{x + \Delta x - x}{\Delta x + i\Delta y}$$
$$= \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x + i\Delta y} \text{ ....(i)}$$

If we calculate limit along real axis ($\Delta x \to 0$ *and* $\Delta y = 0$) and along imaginary axis ($\Delta y \to 0$ *and* $\Delta x = 0$) we get different values of limit. So $f(z) = x$ is nowhere differentiable in $\mathbb{C}$.
2. $f(z) = |z|$ is differentiable everywhere except $z = 0$, when f is considered as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. But in case when we consider complex differentiability, f is nowhere differentiable for all $z \in \mathbb{C}$.
We have $f(x, y) = \sqrt{x^2 + y^2}$ When we consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ then clearly $f_x = \frac{x}{\sqrt{x^2+y^2}}$,
$f_y = \frac{y}{\sqrt{x^2+y^2}}$ both partial derivatives exist for all $(x, y) \neq (0,0)$.
$f(z) = |z| = \sqrt{x^2 + y^2}$ is differentiable everywhere except $z = 0$, when f is considered as a function from $\mathbb{R}^2 \to \mathbb{R}^2$.
But when we consider complex differentiability
We have $f(z) = u + iv = \sqrt{x^2 + y^2}$
Therefore $u = \sqrt{x^2 + y^2}$ and $v = 0$
Clearly $u_x \neq v_y$ and $u_y \neq -v_x$ for all $z \in \mathbb{C}$
Therefore f is nowhere differentiable when we consider complex differentiability.
3. $f(z) = z|z|$ is differentiable everywhere when f is considered as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. But in case when we consider complex differentiability, f is differentiable only for $z \doteq 0$
We have $f(z) = z|z| = x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$
$f(x, y) = x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$

ISBN: 978-91-988207-0-6

Then clearly, we can see that both partial derivatives f_x *and* f_y exist for all $(x, y) \in \mathbb{R}^2$

But when we consider complex differentiability at $z = 0$ by definition of derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(0) = \lim_{z \to 0} \frac{z|z| - 0}{z - 0}$$

$$f'(0) = 0$$

Hance f is $\mathbb{C}$ differentiable at $z = 0$ and for $z \neq 0$ $u_x \neq v_y$ and $u_y \neq -v_x$ (C-R equations are no satisfied)

❖ REFERENCES:

1. Complex Analysis Third Edition [Dennis G. Zill and Patrick Shanahan] Jones & Bartlett Studen Edition
2. Complex Analysis by Anuradha Gupta – Ane's Student Edition